Math 10A
Worksheet, Discussion #32; Tuesday, 7/31/2018
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1 Determinants and Inverses

1.1 Concepts

1. The **determinant** is defined only for square matrices. It is a scalar. The determinant for a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number $ad - bc$.

   The **inverse** of a matrix is defined only for square matrices. The inverse of A is a matrix B such that $AB = BA = I$, the identity matrix with 1s on the diagonal and 0 everywhere else. The inverse is always unique if it exists. The inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix $B = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. The inverse of a square matrix exists if and only if the determinant is nonzero.

   We can use the inverse to easily solve equations of the form $Ax = b$ where x, b are vectors and A is a matrix because we can write $x = A^{-1}b$ if A is invertible. This always has a **unique** solution if A is invertible. If A is not invertible, this is 0 solutions or ∞ solutions.

   There are 2 ways to calculate the determinant of a 3×3 matrix $\begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}$. The first is expansion along the first row to get the determinant is $b_1 \begin{vmatrix} b_5 & b_6 \\ b_8 & b_9 \end{vmatrix} - b_2 \begin{vmatrix} b_4 & b_6 \\ b_7 & b_9 \end{vmatrix} + b_3 \begin{vmatrix} b_4 & b_5 \\ b_7 & b_8 \end{vmatrix}$. The other is to use the diagonal method to get $b_1b_5b_9 + b_2b_6b_7 + b_3b_4b_8 - b_1b_6b_8 - b_2b_4b_9 - b_3b_5b_7$.

   We can determine the number of solutions to an equation $A\vec{x} = \vec{b}$ by the determinant of A and that is given below.

$\det(A)$	$\neq 0$	$= 0$
Number of Solutions	1	0 or ∞

1.2 Problems

2. True **FALSE** We can take determinants of 2×3 matrices but just haven't learned it yet.

3. True **FALSE** If A is a noninvertible square matrix, then $Ax = b$ may still have a unique solution.

4. True **FALSE** It is possible for $A\vec{x} = \vec{0}$ to have no solutions.

Solution: $\vec{x} = \vec{0}$ is a solution so it cannot have no solutions.

5. **TRUE** False If we know that $A\vec{x} = \vec{b}$ has no solutions, then we know what $\det(A)$ is.

Solution: If it has no solutions, then $\det(A) = 0$.

Solution: If A is not invertible, then $Ax = b$ has 0 or ∞ solutions.

6. True **FALSE** If $\det(A) = 0$, then $Ax = b$ has no solutions.

Solution: It is possible for it to have ∞ solutions.

7. Give a 2×2 matrix with determinant equal to 5. Is it unique?

Solution: This is not unique. One solution is $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$.

8. Find the inverses for the following matrices:

$$\begin{pmatrix} 3 & 5 \\ -4 & -8 \end{pmatrix} \qquad \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 5 \\ -1 & -8 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 2 & 5/4 \\ -1 & -3/4 \end{pmatrix} \qquad \begin{pmatrix} -4/7 & 5/7 \\ 3/7 & -2/7 \end{pmatrix} \qquad \begin{pmatrix} 8/3 & 5/3 \\ -1/3 & -1/3 \end{pmatrix}.$$

9. Find the determinant of $\begin{vmatrix} 3 & 2 & 3 \\ -1 & 5 & -3 \\ 7 & -1 & -1 \end{vmatrix}$.

> **Solution:** One option gives us $3(5\cdot(-1)-(-3)\cdot(-1))-2((-1)\cdot(-1)-(-3)\cdot 7)+3((-1)\cdot(-1)-5\cdot 7)=-170$ and the other way gives us $3\cdot 5\cdot(-1)+2\cdot(-3)\cdot 7+3\cdot(-1)\cdot(-1)-3\cdot(-3)\cdot(-1)-2\cdot(-1)\cdot(-1)-3\cdot 5\cdot 7=-170$.

10. Let $A=\begin{pmatrix} 2 & 4 & -1 \\ 2 & 2 & 5 \\ 6 & 8 & 9 \end{pmatrix}$. What is $\det(A)$? How many solutions that $A\vec{x}=\vec{0}$ have?

> **Solution:** The determinant is $2\cdot 2\cdot 9+4\cdot 5\cdot 6+(-1)\cdot 2\cdot 8-2\cdot 5\cdot 8-4\cdot 2\cdot 9-(-1)\cdot 2\cdot 6=0$. Therefore $A\vec{x}=\vec{0}$ has 0 or ∞ many solutions. But $\vec{x}=\vec{0}$ is a solution so it cannot have 0 solutions, so it has ∞ many solutions.

11. Let $A=\begin{pmatrix} 0 & 1 & 0 \\ 5 & 1 & 3 \\ 2 & 2 & 4 \end{pmatrix}$. What is $\det(A)$? How many solutions does $A\vec{x}=\vec{0}$ have?

> **Solution:** The determinant is $-\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix}=-(5\cdot 4-3\cdot 2)=-14$. Since $\det(A)\neq 0$, this has a unique solution and it is $\vec{x}=\vec{0}$.

12. Find x, y such that $2x+3y=4$ and $x+y=1$.

> **Solution:** Let $A=\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$, then $A\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and hence
>
> $$\begin{pmatrix} x \\ y \end{pmatrix}=A^{-1}\begin{pmatrix} 4 \\ 1 \end{pmatrix}=\frac{1}{2\cdot 1-3\cdot 1}\begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

13. Find the solution to $x+2y=3$ and $4x+5y=6$ using matrix vector form.

Solution: We write it as

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \frac{1}{1 \cdot 5 - 2 \cdot 4} \begin{pmatrix} 5 & -2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

14. Write $x + y = 10, y + z = 5, x + z = -1$ in matrix vector form. How many solutions does it have?

Solution:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ -1 \end{pmatrix}.$$

To determine the number of solutions, we can calculate the determinant and it is $1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 - 0 = 2 \neq 0$. Therefore, there is a unique solution.

15. Find a matrix X such that $\begin{pmatrix} 5 & 13 \\ 3 & 8 \end{pmatrix} X = \begin{pmatrix} 1 & 4 & 1 \\ -1 & 2 & 1 \end{pmatrix}$.

Solution:

$$X = \frac{1}{5 \cdot 8 - 13 \cdot 3} \begin{pmatrix} 8 & -13 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ -1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 21 & 6 & -5 \\ -8 & -2 & 2 \end{pmatrix}$$